APPROXIMATE LINEAR SOLUTIONS OF SOME PLANE GAS DYNAMICAL PROBLEMS

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Let the semispace $X > 0$ be filled with gas. We shall assume for simplicity, that the gas is ideal and that its isentropy exponent is γ.

Pressure P_0 appearing at the boundary at the instant $t = 0$, gives rise to a shock wave. Let us denote the initial density of gas and velocity of sound by ρ_0 and c_0, and these behind the shock wave by ρ and c.

Using the coordinate system in which the unperturbed part of the boundary is at rest, we obtain the following system of linearised equations for pressure perturbation p' and velocity components v_x' and v_y'

$$\frac{\partial p'}{\partial t} + \rho c^2 \left(\frac{\partial v_x'}{\partial X} + \frac{\partial v_y'}{\partial Y}\right) = 0, \qquad \frac{\partial v_x'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial X} = 0, \qquad \frac{\partial v_y'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial Y} = 0 \tag{1}$$

Density perturbations can be eliminated with help of the adiabatic condition

$$\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t}$$

Let the pressure perturbations at the boundary of the gas ($X = 0$) be given in two forms

$$p' = Pe^{ikY} \text{ (problem 1)}, \qquad p' = PJ_0(kct)\, e^{ikY} \quad \text{(problem 2)}$$

Here P is a constant and $J_0(kct)$ is a Bessel function.

Conditions at the shock wave ($X = Vt$) are identical to those in [1]

$$v_y' = -U\frac{\partial \xi}{\partial Y}, \qquad v_x' = \frac{1+\delta}{2\rho_0 D}\,p', \qquad \frac{\partial \xi}{\partial t} = \frac{1-\delta}{2\rho_0 U}\,p' \qquad \left(\delta = \frac{1}{M_0^2},\ M_0 = \frac{D}{c_0}\right)$$

Here U is the velocity of the unperturbed boundary, D is the velocity of the shock wave, V is the velocity of the shock wave referred to the boundary and $\xi(Y, t)$ is the deflection of the shock wave from the plane $X = Vt$.

At the initial moment, the front of the shock wave coincides with the boundary of the gas, hence

$$p' = P, \quad v_y' = 0, \quad v_x' = \frac{1+\delta}{2\rho_0 D} P, \quad \xi = 0 \text{ when } t = 0$$

Conditions are identical for both problems, since $J_0(0) = 1$. Dependence of all the magnitudes on the Y-coordinate is given by the factor e^{ikY}. Let us introduce the following notation

$$p' / \rho c = w, \qquad v_x' = u, \qquad v_y' = -iv$$

and change to new variables

$$kX = x = r \sinh \vartheta, \quad kct = y = r \coth \vartheta, \quad r = \sqrt{y^2 - x^2}, \ \tanh \vartheta = x / y$$

We shall adopt a method of solution described in [1]. Convergence of series appearing in the course of this solution is easily proved. Taking into account all changes resulting from the different boundary conditions we have, for a strong shock wave ($c_0 = 0$), the following results:

1°. For pressure perturbations we have

$$w(r, \vartheta) = w_0 J_0(r) - \sum_{n=1}^{\infty} c_n(\vartheta) J_{2n}(r) \tag{2}$$

where $J_{2n}(r)$ is a Bessel function and $c_n(\theta)$ has the following form.

Problem 1,

$$c_n(\vartheta) = B_n \frac{\sinh 2n\vartheta}{\sinh 2n\vartheta_0} - 2w_0 e^{2n\vartheta}, \qquad B_1 = \frac{4w_0(1+\beta+\beta^2)}{(1-\beta)(1+2\beta\coth 2\theta_0)}$$

$$B_2 = \frac{2\beta(1+2\beta\coth 2\vartheta_0) B_1 + 4w_0\beta(1+\beta)}{(1-\beta)(1+2\beta\coth 4\theta_0)}$$

$$B_{n+1} = \frac{2\beta(1+2\coth 2n\vartheta_0) B_n + (1+\beta)(1-2\beta\coth 2(n-1)\vartheta_0) B_{n-1}}{(1-\beta)(1+2\beta\coth 2(n+1)\theta_0)} \qquad (n = 2, 3, \ldots)$$

Problem 2

$$c_n(\vartheta) = B_n \frac{\sinh 2n\vartheta}{\sinh 2n\vartheta_0}, \qquad B_1 = \frac{2w_0}{1+2\beta\coth 2\theta_0}$$

$$B_{n+1} = -B_n \frac{1-2\beta\coth 2n\vartheta_0}{1+2\beta\coth 2(n+1)\theta_0} \qquad (n = 1, 2, \ldots)$$

where

$$\beta^2 = \frac{1}{h+1}, \qquad h = \frac{\gamma+1}{\gamma-1}, \qquad w_0 = \frac{P}{\rho c}, \ \tanh \theta_0 = \beta$$

It should be noted that for the problem 2 such γ can be chosen (in fact $\gamma = 1 + 0.4\sqrt{5}$), that all the coefficients in the series (2) become, beginning from c_3, equal to zero.

2°. Amplitude of distortion of the shock wavefront is given by the series

$$\xi(s) = \frac{P\sqrt{h}}{kP_0} \sum_{n=1}^{\infty} D_n J_{2n-1}(s) \tag{3}$$

and the coefficients D_n are found from

Problem 1 $\quad D_1 = 1, \ \left[1 + 2\beta \frac{\alpha^{2n}+1}{\alpha^{2n}-1}\right] D_{n+1} + \left[1 - 2\beta \frac{\alpha^{2n}+1}{\alpha^{2n}-1}\right] D_n =$

$$= 16 \frac{\alpha^n}{\alpha^{2n} - 1} \qquad \left(\alpha = \frac{1+\beta}{1-\beta}\right) \qquad (n = 1, \ldots)$$

Problem 2

$$D_1 = 1, \qquad D_n = -D_{n-1} \frac{1 - 2\beta \coth 2n\vartheta_0}{1 + 2\beta \coth 2n\vartheta_0} \qquad (n = 2, 3, \ldots)$$

Argument s is given by

$$s = kct\sqrt{1 - \beta^2} = \frac{kL}{\sqrt{h}} = kL_1$$

where L is the distance traversed by the shock wave through the cold medium, while L_1 is the distance covered by the sound signal along the shock wavefront.

3°. Integrating the second equation of (1) twice and taking initial conditions into account, we obtain the relationship between the amplitude of distortion of the boundary, and the time

$$a(r) = \frac{\sqrt{h+1}}{2kc} w_0 r + \frac{1}{kc} \sum_{n=1}^{\infty} c_n' \sum_{k=n}^{\infty} \left(-J_{2k+1}(r) + 2\sum_{l=k}^{\infty} J_{2l+1}(r)\right) \qquad (4)$$

Here

$$r = kct, \qquad w_0 = \frac{P}{\rho c}, \qquad c_n' = \left(\frac{1}{n}\frac{dc_n(\vartheta)}{d\vartheta}\right)_{\vartheta=0}$$

It can easily be established from (3) and (4) that as $L \to \infty$, amplitude of distortion of the shock wavefront decays as $s^{-1/2}$, while the amplitude of distortion of the boundary increases linearly with time.

Asymptotic behavior of the magnitude $a(r)$ is identical to that which would take place, if the pressure at the boundary $p'(X = 0)$ was, in problem 2, also independent to time and if the perturbations developed independently at the separate segments.

BIBLIOGRAPHY

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